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Attraction of the Homogeneous Spherical Segment.[*]

BY G. W. HILL.

SECTION I.—*Reduction of the Attraction to Single Quadratures.*

We may suppose the segment treated does not exceed the hemisphere. The centre of the sphere may be adopted as the origin of a system of rectangular coordinates. The straight line going from the centre to the attracted point may be taken as the axis of x; in case this line is indeterminate, the axis of the segment may be taken. The axis of y is put in the plane which contains the axes of x and of the segment; its positive direction is on the side where lies the centre of gravity of the segment. The positive direction of the axis of z is immaterial. In the treatment of the component of the attraction in the direction of x it is convenient to substitute for y and z the variables r and ω such that

$$y = r\cos\omega, \quad z = r\sin\omega.$$

All radicals occuring in the following expressions are to be taken positively.

Denoting the density by ρ, and, for brevity, putting

$$b = x - x',$$

the components of the attraction, severally in the directions of the axes of x and y, are

$$X = \rho\int\int\int\frac{b}{[b^2 + r^2]^{\frac{3}{2}}}r dr\, d\omega\, dx, \quad Y = \rho\int\int\int\frac{y}{[b^2 + y^2 + z^2]^{\frac{3}{2}}}dy\, dz\, dx.$$

There is no need of mentioning Z as it plainly vanishes. The integrations are understood to be taken in the order of the written differentials.

The equation of the sphere to which the segment belongs is

$$x^2 + y^2 + z^2 = x^2 + r^2 = a^2,$$

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[*] A possible application in geodesy induced the composition of this memoir. Although the simple case where the attracted point lies on the axis of the segment has been several times treated, after a long search, no general investigation could be found. Fearing the matter might have escaped me, I consulted the bibliographical knowledge of mathematical friends, but with only a negative result.

and that of the plane, part of which forms the base of the segment, may be written

$$x\cos\phi + y\sin\phi = a\cos\gamma,$$

where γ is the angular radius of the segment, always in the first quadrant, and ϕ is the angle between the axes of the segment and of x, always less than two right angles. The triple integrals must then be extended so as to cover all points satisfying the two inequations

$$x^2 + y^2 + z^2 < a^2, \qquad x\cos\phi + y\sin\phi > a\cos\gamma.$$

The indefinite integrals of the expressions under the signs of integration in X and Y, the first with reference to r, the second with reference to y, are severally

$$-\frac{b}{\sqrt{b^2 + r^2}} \text{ and } -\frac{1}{\sqrt{b^2 + y^2 + z^2}}.$$

The section of the segment by the plane perpendicular to the the axis of x is discontinuous at the two points

$$x = a\cos(\phi + \gamma) = \mathrm{b}, \qquad x = a\cos(\phi - \gamma) = \mathrm{a}.$$

Therefore, in the derivation of X, we are forced to divide the integration with respect to x into two parts, and to establish three distinct cases. Let us, for brevity, put

$$c = \sqrt{a^2 - x^2}, \qquad d\sin\phi = a\cos\gamma - x\cos\phi.$$

In the first part, the integration is from $\omega = 0$ to $\omega = 2\pi$, and, in the second part, from $r = \frac{d}{\cos\omega}$ to $r = c$. We therefore put

$$A = 2\pi\rho\int\int\frac{b}{[b^2 + r^2]^{\frac{3}{2}}}r\,dr\,dx, \qquad B = \rho\int\int\left[\frac{b\cos\omega}{\sqrt{d^2 + b^2\cos^2\omega}} - \frac{b}{\sqrt{b^2 + c^2}}\right]d\omega\,dx.$$

The statement of the three cases is as follows:

CASE I.—Where $\phi + \gamma > 180°$.

$$X = A + B.$$

CASE II.—Where $\phi - \gamma$ and $\phi + \gamma$ both lie between $0°$ and $180°$.

$$X = B.$$

CASE III.—Where $\phi - \gamma$ is a negative angle.

$$X = A + B.$$

Case III differs from Case I in that the limits of integration for A are different.

In Y the integration with respect to y is from $y=d$ to $y=\sqrt{c^2-z^2}$, and, in every case

$$Y=\rho\int\int\left[\frac{1}{\sqrt{b^2+d^2+z^2}}-\frac{1}{\sqrt{b^2+c^2}}\right]dz\,dx.$$

Proceeding to remove another sign of integration from these expressions, we note the indefinite integrals of the quantities under the signs of integration in X are severally

$$-\frac{b}{\sqrt{b^2+r^2}}\text{ and arc sin}\frac{b\sin\omega}{\sqrt{b^2+d^2}}-\frac{b}{\sqrt{b^2+c^2}}\omega.$$

The first must be taken between the limits $r=0$ and $r=c$, the second from $\omega=-\text{arc sin}\dfrac{\sqrt{c^2-d^2}}{c}$ to $\omega=\text{arc sin}\dfrac{\sqrt{c^2-d^2}}{c}$. Then, altering the mode of expression of A and B, we have

$$A=2\pi\rho\int\left[\frac{b}{\sqrt{b^2}}-\frac{b}{\sqrt{b^2+c^2}}\right]dx,$$

$$B=2\rho\int\left[\text{arc sin}\left(\frac{b}{c}\,\frac{\sqrt{c^2-d^2}}{\sqrt{b^2+d^2}}\right)-\frac{b}{\sqrt{b^2+c^2}}\text{arc sin}\,\frac{\sqrt{c^2-d^2}}{c}\right]dx,$$

and the three cases are, as before,

CASE I.— $X=A+B$. CASE II.— $X=B$. CASE III.— $X=A+B$.

In the expression for A, it has been preferred to write $\dfrac{b}{\sqrt{b^2}}$ instead of ± 1, as the latter notation involves the necessity of explaining the ambiguous sign; and the arcs in B are supposed to be taken between the limits $\pm\dfrac{\pi}{2}$.

The integration in Y with respect to z is from $z=-\sqrt{c^2-d^2}$ to $z=\sqrt{c^2-d^2}$, and thus

$$Y=\rho\int\left[\log\frac{\sqrt{b^2+c^2}+\sqrt{c^2-d^2}}{\sqrt{b^2+c^2}-\sqrt{c^2-d^2}}-2\frac{\sqrt{c^2-d^2}}{\sqrt{b^2+c^2}}\right]dx.$$

The components of the attraction are now reduced to single quadratures. The limits of integration for B and Y are from $x=\mathrm{b}$ to $x=\mathrm{a}$; for the A of Case I, from $x=-a$ to $x=\mathrm{b}$: for the A of Case III, from $x=\mathrm{a}$ to $x=a$.

The integration indicated in the expression A is easily accomplished; for Case I we have

$$A=2\pi\rho\left[\sqrt{(\mathrm{b}-x')^2}-\frac{x'^2+a^2-x'a}{3x'^2}(a+x')-\frac{2x'^2-a^2-x'\mathrm{b}}{3x'^2}\sqrt{x'^2+a^2-2x'\mathrm{b}}\right];$$

and for Case III we have

$$A = 2\pi\rho\left[-\sqrt{(\mathrm{a}-x')^2}+\frac{x'^2+a^2+x'a}{3x'^2}\sqrt{(a-x')^2}+\frac{2x'^2-a^2-x'a}{3x'^2}\sqrt{x'^2+a^2-2x'\mathrm{a}}\right]$$

SECTION II.—*Reduction of the Attractions to the Canonical Forms for Elliptic Integrals.*

The definite integrals, just obtained as the values of the functions B and Y, have the great advantage of having no discontinuities for the case when the attracted point is on the edge of the segment, or when $x' = a$, $\phi = \gamma$. Still, if we would learn the essential qualities of these functions, the arcs must be made to disappear from under the signs of integration, and their places taken by purely algebraic expressions. For accomplishing this, a ready method is that called integration by parts, the arcs momentarily being regarded as constant. Thus discontinuities are introduced into the values of the definite integrals when the elements of the problem show that the attracted point is at the edge of the segment. However, as we know the components of the attraction are finite in this case, the sum of the discontinuities must vanish. Thus is suggested a method of escaping their consideration. By neglecting every discontinuity as it presents itself, we must necessarily arrive at the correct value of the component. But this course of proceeding does not remove all the difficulties. The general expressions for the components involve terms formed by the product of two factors, of which the first is a multiplier of the second a definite integral. In case the attracted point is at the edge of the segment, the factor vanishes, while the definite integral becomes infinite. It is not possible to remove the factor within the sign of integration as it is variable. The only course to be pursued is to put the factor in the form $A\varepsilon$, and the value of the definite integral in the form $\frac{B}{\varepsilon}$, where ε is an infinitesimal expressing the distance of the elements of the problem from the singular point; then the value of the term is evidently AB. But the determination of A and B is sometimes baffling, especially when, for the purpose of removing imaginary parameters, definite integrals, whose values go at a bound from 0 to ∞, are introduced.

Attending to B and regarding only the first term, after integration by parts, we have the term

$$\left[2\rho x \text{ arc sin}\left(\frac{b}{c}\,\frac{\sqrt{c^2-d^2}}{\sqrt{b^2+d^2}}\right)\right]_{\mathrm{b}}^{\mathrm{a}}.$$

It is plain this expression vanishes except on three occasions; first, when $\phi = \gamma$; second, when $\phi = 180^\circ - \gamma$; and, third, when both these conditions are fulfilled, or when $\phi = \gamma = 90^\circ$. In the first case its value is

$$2\rho a \text{ arc sin} \frac{a - x'}{\sqrt{(a - x')^2}},$$

which is $\pi\rho a$, if $a - x'$ is positive, but $-\pi\rho a$, if the same quantity is negative. In the second case its value is

$$2\rho a \text{ arc sin} \left[\frac{-a - x'}{\sqrt{2a}} \sqrt{\frac{2a}{(a + x')^2}} \right] = -\pi\rho a.$$

In the third case the sum of these values, or 0, if $a - x'$ is positive, or $-2\pi\rho a$, if it is negative.

The second term of B, after the integration by parts, gives rise to the term

$$2\rho \left[-\frac{2x'^2 - a^2 - x'x}{3x'^2} \sqrt{b^2 + c^2} \text{ arc sin} \frac{\sqrt{c^2 - d^2}}{c} \right]_{\mathrm{b}}^{\mathrm{a}}.$$

This term vanishes in all cases except when $a \cos \gamma - \mathrm{a} \cos \phi = 0$, or $a \cos \gamma - \mathrm{b} \cos \phi = 0$, or the same conditions as obtained in the first term. Therefore, in the first case, the value of this term is

$$-\pi\rho \frac{(2x' + a)(x' - a)}{3x'^2} \sqrt{(x' - a)^2},$$

in the second case,

$$\pi\rho \frac{(2x' - a)(x' + a)^2}{3x'^2},$$

and, in the third case, the sum of these,

$$\frac{2}{3} \pi\rho a \frac{x'^2 - a^2}{x'^2} \text{ or } \frac{2}{3} \pi\rho (2x' - a),$$

according as $x' - a$ is positive or negative.

Next attending to Y, if we put

$$P = \frac{\sqrt{c^2 - d^2}}{\sqrt{b^2 + c^2}} = \frac{\sqrt{(\mathrm{a} - x)(x - \mathrm{b})}}{\sin \phi \sqrt{x'^2 + a^2 - 2x'x}},$$

we have

$$Y = \rho \int_{\mathrm{b}}^{\mathrm{a}} \left[\log \frac{1 + P}{1 - P} - 2P \right] dx,$$

and integration by parts gives the term

$$\left[\rho x \left(\log \frac{1 + P}{1 - P} - 2P \right) \right]_{\mathrm{b}}^{\mathrm{a}}.$$

P always vanishes at both limits, unless at the upper limit when $x' = a$ and $\phi = \gamma$; it cannot vanish at the lower limit as this would necessitate $x' = -a$, which cannot occur with our conventions. Here we are beset with difficulties, but, it seems in this case that we are warranted in regarding $\sqrt{b^2 + c^2}$ as an infinitesimal of a lower order than $\sqrt{c^2 - d^2}$. Thus we shall say that the term introduced by the integration by parts, and which is free from the integrating sign, vanishes in all cases.

Neglecting the discontinuities, and calling B, thus modified, X', we have

$$X' = -2\rho \int_{\mathrm{b}}^{\mathrm{a}} \left\{ x \frac{d}{dx} \left[\text{arc sin} \left(\frac{b}{c} \frac{\sqrt{c^2 - d^2}}{\sqrt{b^2 + d^2}} \right) \right] - \frac{2x'^2 - a^2 - x'x}{3x'^2} \right.$$
$$\left. \times \sqrt{b^2 + c^2} \frac{d}{dx} \left[\text{arc sin} \frac{\sqrt{c^2 - d^2}}{c} \right] \right\} dx,$$

$$Y = -2\rho \int_{\mathrm{b}}^{\mathrm{a}} \left[P + \frac{x}{1 - P^2} \frac{dP}{dx} \right] dx.$$

Then the expressions for X, in its three cases, are

CASE I. $X = 2\pi\rho \left[\pm (x' - \mathrm{b}) - \frac{a^3 + x'^3}{3x'^2} \right] + X'$.
CASE II. $X = X'$.
CASE III. $X = 2\pi\rho \left[\mp (\mathrm{a} - x') \pm \frac{a^3 - x'^3}{3x'^2} \right] + X'$.

Here the ambiguous sign in Case I must be so read that the quantity may be positive, and in Case III, of the two ambiguous signs, the first must be so read that the quantity may be negative, and the second so read that the term may be positive.

Performing the differentiation with respect to x, indicated in the expression for X', and writing f for $x' \cos\phi - a \cos\gamma$, the first term, after some reductions takes the form

$$2\rho \int_{\mathrm{b}}^{\mathrm{a}} \left[\frac{d \sin\phi\, (bx + c^2)\, x}{c^2} + f \frac{b^2 + c^2}{b^2 + d^2} x \right] \frac{dx}{\sin\phi \sqrt{b^2 + c^2} \sqrt{c^2 - d^2}}.$$

Unless there are discontinuities at the limits, for the factor x, which multiplies the expression under the sign of integration, we may substitute $x - \mathrm{b}$, as the value of the definite integral is not thereby altered.

The second term of X' takes the form

$$2\rho \int_{\mathrm{b}}^{\mathrm{a}} \frac{a\,(2x'^2 - a^2 - x'x)\,(a \cos\phi - x \cos\gamma)\,(b^2 + c^2)}{3x'^2\, c^2} \frac{dx}{\sin\phi \sqrt{b^2 + c^2} \sqrt{c^2 - d^2}}.$$

At this point it is desirable to substitute a new variable θ for x, such that

$$x = \mathrm{b} + (\mathrm{a} - \mathrm{b}) \sin^2\theta = a \cos(\phi + \gamma) + 2a \sin\phi \sin\gamma \sin^2\theta.$$

The limits of integration with respect to θ are from 0 to $\frac{\pi}{2}$. If we adopt two constants k and g, such that

$$k^2 = \frac{2(\mathrm{a} - \mathrm{b})x'}{x'^2 + a^2 - 2x'\mathrm{b}} = \frac{4ax' \sin\phi \sin\gamma}{x'^2 + a^2 - 2x'\mathrm{b}},$$

$$g = \frac{\mathrm{a} - \mathrm{b}}{\sin\phi \sqrt{x'^2 + a^2 - 2x'\mathrm{b}}} = \frac{2a \sin\gamma}{\sqrt{x'^2 + a^2 - 2x'\mathrm{b}}},$$

k will be the modulus of the elliptic integrals involved in the problem, and, according to the usual notation, we put $\sqrt{1 - k^2 \sin^2\theta} = \Delta$. Then

$$\frac{dx}{\sin\phi \sqrt{b^2 + c^2}\sqrt{c^2 - d^2}} = \frac{g}{2a \sin\gamma} \frac{d\theta}{\Delta},$$

and

$$X' = \frac{2\rho g}{a \sin\gamma} \int_0^{\frac{\pi}{2}} \left[\frac{(a\cos\gamma - x\cos\phi)(a^2 - x'x)(x - \mathrm{b}) + \frac{a}{3x'^2}(2x'^2 - a^2 - x'x)(a\cos\phi - x\cos\gamma)(b^2 + c^2)}{c^2} \right.$$

$$\left. + \frac{f(x - \mathrm{b})}{1 - P^2} \right] \frac{d\theta}{\Delta} = \frac{2\rho g}{a \sin\gamma} \int_0^{\frac{\pi}{2}} \left[\frac{Q}{c^2} + f \frac{b^2 + c^2}{b^2 + d^2}(x - \mathrm{b}) \right] \frac{d\theta}{\Delta}.$$

We can suppose

$$\frac{Q}{c^2} = A + A'x + \frac{A''}{a - x} + \frac{A'''}{a + x}.$$

Then

$$(A + A'x)(a^2 - x^2) + A''(a + x) + A'''(a - x) = Q.$$

Making $x = a$, we get

$$2aA'' = 2a^3 \sin^2\frac{\phi + \gamma}{2}(\cos\gamma - \cos\phi)(a - x') + \frac{a^2}{3x'^2}(2x' + a)(x' - a)^3(\cos\phi - \cos\gamma).$$

Making $x = -a$, we get

$$2aA''' = -2a^3 \cos^2\frac{\phi + \gamma}{2}(\cos\gamma + \cos\phi)(x' + a) + \frac{a^2}{3x'^2}(2x' - a)(x' + a)^3(\cos\phi + \cos\gamma).$$

Making $x = 0$, we get

$$a^2 A + a(A'' + A''') = -a^3 \mathrm{b} \cos\gamma + \frac{a^2 \cos\phi}{3x'^2}(2x'^2 - a^2)(x'^2 + a^2).$$

A' is derived by making the division. Thus are obtained the following values:

$$\begin{aligned}
A \ &= [\tfrac{1}{3} a + x' \cos (\phi + \gamma)]\, a \cos \phi - \frac{a}{3x'} (2x'^2 - a^2) \cos \gamma, \\
A' \ &= - x' \cos \phi + \tfrac{2}{3} a \cos \gamma, \\
A'' &= a^2 (x' - a) \left[\sin^2 \frac{\phi + \gamma}{2} + \frac{(2x' + a)(x' - a)^2}{6ax'^2} \right] (\cos \phi - \cos \gamma), \\
A''' &= - a^2 (x' + a) \left[\cos^2 \frac{\phi + \gamma}{2} - \frac{(2x' - a)(x' + a)^2}{6ax'^2} \right] (\cos \phi + \cos \gamma).
\end{aligned}$$

In the second portion of X' we have

$$P = \frac{g \sin \theta \cos \theta}{\Delta}.$$

Thus

$$\begin{aligned}
\frac{f(x - \mathrm{b})}{1 - P^2} &= 2fa \sin \phi \sin \gamma \frac{(1 - k^2 \sin^2 \theta) \sin^2 \theta}{1 - (g^2 + k^2) \sin^2 \theta + g^2 \sin^4 \theta} \\
&= - 2fx' \sin^2 \phi + 2fa \sin \phi \sin \gamma \frac{\dfrac{k^2}{g^2} + \left(k'^2 - \dfrac{k^4}{g^2} \right) \sin^2 \theta}{1 - (g^2 + k^2) \sin^2 \theta + g^2 \sin^4 \theta},
\end{aligned}$$

where $k'^2 = 1 - k^2$.

By putting

$$n = - \frac{\sin \phi \sin \gamma}{\sin^2 \dfrac{\phi + \gamma}{2}}, \qquad n' = \frac{\sin \phi \sin \gamma}{\cos^2 \dfrac{\phi + \gamma}{2}},$$

$$\begin{aligned}
B \ &= \frac{1}{a} \left[\frac{1}{3} a^2 - 2x'^2 \sin^2 \phi \right] \cos \phi - \frac{1}{3x'} [2x'^2 (1 - 3 \sin^2 \phi) \\
&\qquad\qquad - 2ax' \cos (\phi + \gamma) - a^2] \cos \gamma, \\
B' \ &= 2 \left[- x' \cos \phi + \frac{2}{3} a \cos \gamma \right] \sin \phi \sin \gamma, \\
B'' &= - (x' - a) \left[\sin^2 \frac{\phi + \gamma}{2} + \frac{(2x' + a)(x' - a)^2}{6ax'^2} \right] \frac{\sin \dfrac{\phi - \gamma}{2}}{\sin \dfrac{\phi + \gamma}{2}}, \\
B''' &= - (x' + a) \left[\cos^2 \frac{\phi + \gamma}{2} - \frac{(2x' - a)(x' + a)^2}{6ax'^2} \right] \frac{\cos \dfrac{\phi - \gamma}{2}}{\cos \dfrac{\phi + \gamma}{2}},
\end{aligned}$$

we have the expression

$$\begin{aligned}
X' = \frac{2\rho g}{\sin \gamma} \int_0^{\frac{\pi}{2}} \Bigg[& B + B' \sin^2 \theta + \frac{B''}{1 + n \sin^2 \theta} + \frac{B'''}{1 + n' \sin^2 \theta} \\
& + 2f \sin \phi \sin \gamma \frac{\dfrac{k^2}{g^2} + \left(k'^2 - \dfrac{k^4}{g^2} \right) \sin^2 \theta}{1 - (g^2 + k^2) \sin^2 \theta + g^2 \sin^4 \theta} \Bigg] \frac{d\theta}{\Delta}.
\end{aligned}$$

We next reduce Y to a similar form. The equivalent for this may be written

$$Y = -2\rho \int_0^{\frac{\pi}{2}} \left[P \frac{dx}{d\theta} + \frac{x - \mathrm{b}}{1 - P^2} \frac{dP}{d\theta} \right] d\theta.$$

We have

$$\frac{dP}{d\theta} = g \frac{1 - 2\sin^2\theta + k^2 \sin^4\theta}{\Delta^3}.$$

Thus

$$Y = -4\rho a g \sin\phi \sin\gamma \int_0^{\frac{\pi}{2}} \left[2\sin^2\theta\cos^2\theta + \frac{(1 - 2\sin^2\theta + k^2\sin^4\theta)\sin^2\theta}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta} \right] \frac{d\theta}{\Delta}.$$

SECTION III.—*Removal of the Imaginary Parameters.*

The expressions just derived for X' and Y involve the divisor

$$1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta.$$

The values of k and g are such as make the two parameters resulting from this divisor imaginary. Legendre [*] has shown how these parameters may be replaced by a single real parameter; but his method is laborious and necessitates the computation of many constants, needless in our case. Fortunately the two integrals concerned are complete, and the modulus k and the two imaginary parameters n and n' satisfy the relation

$$(1 - k)(1 + k) = (1 + n)(1 + n').$$

These circumstances enable us to reduce the elaboration very much.

In the first place let

$$p = \frac{\sin\theta\cos\theta}{\Delta}.$$

Then, by differentiation,

$$\frac{dp}{1 - g^2p^2} = \frac{1 - 2\sin^2\theta + k^2\sin^4\theta}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta} \frac{d\theta}{\Delta}.\dagger$$

Integrating between the limits 0 and $\frac{\pi}{2}$,

$$\int_0^{\frac{\pi}{2}} \frac{1 - 2\sin^2\theta + k^2\sin^4\theta}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta} \frac{d\theta}{\Delta} = 0.$$

However, it must be noted that there is a discontinuity in the value of this definite integral when $k = 1$. It may be multiplied by a constant at our disposal, and the product joined to either of the previous expressions for X' and Y.

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[*] Traité des Fonctions Elliptiques, Tom. I, p. 143.

[†] Traité des Fonctions Elliptiques, Tom. I, p. 72.

In the second place, ζ being a constant, let

$$p=\frac{\sin\theta\cos\theta}{(1+\zeta\sin^2\theta)\,\Delta}\cdot$$

It is plain that, since $\sin\theta\cos\theta$ vanishes at both limits of the integration,

$$\int_0^{\frac{\pi}{2}}\frac{\frac{dp}{d\theta}}{1+\varkappa p^2}\,d\theta=0,$$

$\varkappa$ being a constant at our disposal. The differentiation being performed,

$$\frac{dp}{1+\varkappa p^2}=\frac{1-(\zeta+2)\sin^2\theta+(2\zeta+1)k^2\sin^4\theta-\zeta k^2\sin^6\theta}{(1+\zeta\sin^2\theta)^2(1-k^2\sin^2\theta)+\varkappa\sin^2\theta\,(1-\sin^2\theta)}\,\frac{d\theta}{\Delta}\cdot$$

If $\varkappa$ is rightly assumed, the denominator of the first factor of the right member of this can be put under the form

$$[1-(g^2+k^2)\sin^2\theta+g^2\sin^4\theta]\,(1+m\sin^2\theta),$$

provided also that ζ has the value

$$\zeta=-\frac{2g^2}{g^2+k^2}\cdot$$

We have no need to know the value of $\varkappa$, but that of m is

$$m=-\frac{4k^2g^2}{(g^2+k^2)^2}\quad\text{or}\quad 1+m=\left(\frac{g^2-k^2}{g^2+k^2}\right)^2.$$

It is evident now that we may form the definite integral

$$\int_0^{\frac{\pi}{2}}\left[\frac{1}{\zeta}+\frac{A+A'\sin^2\theta}{1-(g^2+k^2)\sin^2\theta+g^2\sin^4\theta}+\frac{B}{1+m\sin^2\theta}\right]\frac{d\theta}{\Delta}=0.$$

We may readily satisfy ourselves that the values of A, A' and B are

$$A=1+\frac{k^2}{g^2},\qquad A'=-(g^2+k^2)\frac{k^2}{g^2},\qquad B=\tfrac{1}{2}\left(1-\frac{k^2}{g^2}\right).^*$$

As in the former, there is a discontinuity in the value of this definite integral, when $k=1$. It may be multiplied by a constant at our disposal, and the product joined to either of the preceding expressions for X' and Y.

It is evident that to these constants (four in number) we may give such values that the term under the sign of integration, having the divisor

$$1-(g^2+k^2)\sin^2\theta+g^2\sin^4\theta,$$

will vanish in both cases, and be replaced by terms having the divisor

$$1+m\sin^2\theta.$$

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[*] I am greatly indebted to Cayley's Elementary Treatise on Elliptic Functions, pp. 127-130, for making the road clear to these values.

Attending first to the modification of X', we may write

$$X' = 2\rho g \sin\phi \int_0^{\frac{\pi}{2}} \left[\Theta + 2f \frac{\dfrac{k^2}{g^2} + \left(k'^2 - \dfrac{k^4}{g^2}\right)\sin^2\theta}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta} \right] \frac{d\theta}{\Delta}.$$

Multiply the first definite integral of zero value by f, and the second by $-f$, and add the products to the quantity under the integrating sign. As the result we get

$$X' = 2\rho g \sin\phi \int_0^{\frac{\pi}{2}} \left[\Theta + f\left(\frac{1}{\zeta} - \frac{k^2}{g^2}\right) + \frac{\frac{1}{2}f\left(1 - \dfrac{k^2}{g^2}\right)}{1 + m\sin^2\theta} \right] \frac{d\theta}{\Delta}.$$

We can simplify the second definite integral by dividing it by $-\dfrac{1}{\zeta}$, and thus

$$\int_0^{\frac{\pi}{2}} \left[-1 + 2\frac{1 - k^2\sin^2\theta}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta} + \frac{\sqrt{1+m}}{1 + m\sin^2\theta} \right] \frac{d\theta}{\Delta} = 0,$$

where $\sqrt{1+m}$ is to be taken positively. If the first definite integral is multiplied by h, and the second, just given, by H, constants at our disposal, and the products added to the quantity under the integrating sign in the value of Y, we have

$$\int_0^{\frac{\pi}{2}} \left[\frac{(1 - 2\sin^2\theta + k^2\sin^4\theta)(h + \sin^2\theta) + 2H(1 - k^2\sin^2\theta)}{1 - (g^2 + k^2)\sin^2\theta + g^2\sin^4\theta} + \frac{H\sqrt{1+m}}{1 + m\sin^2\theta} \right.$$
$$\left. - H + 2\sin^2\theta\cos^2\theta \right] \frac{d\theta}{\Delta}$$

Here h and H must be so chosen that the first fraction of this shall become integral. Then it is plain the expression will assume the form

$$\int_0^{\frac{\pi}{2}} \left[h + H + \left(2 + \frac{k^2}{g^2}\right)\sin^2\theta - 2\sin^4\theta + \frac{H\sqrt{1+m}}{1 + m\sin^2\theta} \right] \frac{d\theta}{\Delta}.$$

The equations, which determine h and H, are

$$g^2(g^2 + k^2 - 2)h + 2g^4 H = k^2 - g^2,$$
$$g^2(g^2 - k^2)h + 2g^4 H = k^2(g^2 + k^2) - 2g^2,$$

whence are derived

$$h = -\frac{g^2 + k^2}{2g^2}, \qquad H = \frac{1}{4g^4}\left[(g^2 + k^2)^2 - 4g^2\right].$$

Thus

$$Y = -4\rho a g \sin\phi \sin\gamma \int_0^{\frac{\pi}{2}} \left[\frac{1}{4g^4}(g^4 - k^4 + 4g^2) + \left(2 + \frac{k^2}{g^2}\right)\sin^2\theta - 2\sin^4\theta \right.$$
$$\left. + \frac{H\sqrt{1+m}}{1 + m\sin^2\theta} \right] \frac{d\theta}{\Delta}.$$

SECTION IV.—*Evaluation of the Elliptic Integrals by Means of the Θ Function.*

The readiest method of arriving at the numerical values of X' and Y is by the use of the Θ function of Jacobi. For the parts which involve the first and second complete functions of Legendre, we have

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\Delta} = K, \quad \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\Delta} d\theta = \frac{1}{k^2}(K - E'),$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{\Delta} d\theta = \frac{1}{3k^2}(2K - E') - \frac{2}{3k^4}(K - E').$$

K and E' can be derived from the equations

$$k = \sin\psi, \quad \frac{q + q^9 + q^{25} + \dots}{1 + 2(q^4 + q^{16} + q^{36} + \dots)} = \left(\frac{\sin\frac{1}{2}\psi}{1 + \sqrt{\cos\psi}}\right)^2, \quad \sqrt{\cos\psi} = \chi,$$

$$q = \frac{1}{2}\frac{1-\chi}{1+\chi} + 2\left(\frac{1}{2}\frac{1-\chi}{1+\chi}\right)^5 + 30\left(\frac{1}{2}\frac{1-\chi}{1+\chi}\right)^9 + \dots,$$

$$K = \frac{\pi}{2}\left(\frac{2}{1+\sqrt{\cos\psi}}\right)^2 [1 + 2(q^4 + q^{16} + q^{36} + \dots)]^2,$$

$$K - E' = \frac{\pi}{2}\,\frac{(1 + \sqrt{\cos\psi})^3}{\sqrt{\cos\psi}}\,\frac{q - 4q^4 + 9q^9 - 16q^{16} + \dots}{[1 + 2(q^4 + q^{16} + q^{36} + \dots)]^3}.$$

These formulæ have the inconvenience of approximating the indeterminate form $\frac{0}{0}$ as k becomes smaller. This may be removed by employing the relation

$$k^2 = 16q\left[\frac{1 + q^2 + q^6 + q^{12} + \dots}{1 + 2(q + q^4 + q^9 + \dots)}\right]^4.$$

Consequently

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\Delta} d\theta = \frac{\pi}{32}\,\frac{(1 + \sqrt{\cos\psi})^3}{\sqrt{\cos\psi}}\,\frac{1 - 4q^3 + 9q^8 - 16q^{15} + \dots}{[1 + 2(q^4 + q^{16} + q^{36} + \dots)]^3}$$
$$\times\left[\frac{1 + 2(q + q^4 + q^9 + \dots}{1 + q^2 + q^6 + q^{12} + \dots)}\right]^4.$$

In case k is quite small, we may employ the series in powers of k,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{\Delta} d\theta = \frac{\pi}{6}\left[\left(\frac{1}{2}\right)^2\frac{3}{2} + \left(\frac{1.3}{2.4}\right)^2\frac{4}{3}k^2 + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{5}{4}k^4 + \dots\right].$$

The similar series for the second definite integral is

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\Delta} d\theta = \frac{\pi}{2}\left[\frac{1}{2} + \left(\frac{1}{2}\right)^2\frac{3}{4}k^2 + \left(\frac{1.3}{2.4}\right)^2\frac{5}{6}k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{7}{8}k^6 + \dots\right].$$

The complete elliptic integrals of the third kind of Legendre have the general form

$$\int_0^{\frac{\pi}{2}} \frac{A}{1 + n \sin^2 \theta} \frac{d\theta}{\Delta},$$

where n may be positive or negative. Consider first the case where it is negative. Here, if k^2 exceeds $-n$, we derive a from the equation (using the Gudermannian notation)

$$\operatorname{sn}(a) = \frac{\sqrt{-n}}{k},$$

and we have

$$\int_0^{\frac{\pi}{2}} \frac{A}{1 + n \sin^2 \theta} \frac{d\theta}{\Delta} = AK\left[1 + \sqrt{\frac{-n}{(k^2 + n)(1 + n)}} \frac{\Theta'(a)}{\Theta(a)}\right],$$

where

$$Z(a) = \frac{\Theta'(a)}{\Theta(a)} = \frac{\pi}{2K} \frac{4q \sin\left(\frac{\pi}{K} a\right) - 8q^4 \sin 2\left(\frac{\pi}{K} a\right) + 12q^9 \sin 3\left(\frac{\pi}{K} a\right) - \dots}{1 - 2q \cos\left(\frac{\pi}{K} a\right) + 2q^4 \cos 2\left(\frac{\pi}{K} a\right) - 2q^9 \cos 3\left(\frac{\pi}{K} a\right) + \dots}.\text{*}$$

If either one or both of $-n$, k are unity, this definite integral is infinite. But, as the attraction of the segment cannot be infinite, it follows that the multiplier A must, in this case, vanish, and the product be finite.

If n is positive we derive a from the equation (i is put for $\sqrt{-1}$)

$$\operatorname{sn}(ia) = \frac{\sqrt{-n}}{k},$$

and we have

$$\int_0^{\frac{\pi}{2}} \frac{A}{1 + n \sin^2 \theta} \frac{d\theta}{\Delta} = AK\left[1 + i\sqrt{\frac{n}{(k^2 + n)(1 + n)}} \frac{\Theta'(ia)}{\Theta(ia)}\right].$$

But

$$i\frac{\Theta'(ia)}{\Theta(ia)} = -\frac{\pi}{K}\left[\frac{q^{1-b}}{1 - q^{1-b}} - \frac{q^{1+b}}{1 - q^{1+b}} + \frac{q^{3-b}}{1 - q^{3-b}} - \frac{q^{3+b}}{1 - q^{3+b}} + \dots\right],$$

where $b = \frac{a}{K'}$, K' being the complementary period.†

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* Jacobi, Fundamenta Nova, p. 184.

† Journal für die Mathematik, Vol. XXXIX, p. 345.

The third case where $-n$ exceeds k^2 can only have place for the parameters n and m. If either of the equations

$$1+n=\frac{\sin^2\frac{\phi-\gamma}{2}}{\sin^2\frac{\phi+\gamma}{2}}=k'^2\sin^2\lambda, \qquad 1+m=\left(\frac{g^2-k^2}{g^2+k^2}\right)^2=k'^2\sin^2\lambda,$$

affords a real value for λ, the corresponding elliptic integral must be eliminated and replaced by another having a different parameter, by means of the relation

$$\frac{1+n}{n}\int_0^{\frac{\pi}{2}}\frac{1}{1+n\sin^2\theta}\frac{d\theta}{\Delta}-\frac{1-n''}{n''}\int_0^{\frac{\pi}{2}}\frac{1}{1-n''\sin^2\theta}\frac{d\theta}{\Delta}=\frac{k^2}{nn'}K,$$

where n'' is determined by the equation

$$(1+n)(1-n'')=k'^2.^*$$

SECTION V.—*Investigation of the Cases of Indetermination in the Expressions for X' and Y.*

From the value we have given for k^2, it is plain that if, for the moment, M denote the greatest distance of the attracted point from the edge of the segment and N the least, we will have the equation

$$k^2=\frac{M^2-N^2}{M^2}.$$

Thus $k=0$ when the point is on the axis of the segment, and $k=1$ when it is on the edge; and a value intermediate in other situations

The definite integrals

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{\Delta} \text{ and } \int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{\Delta}d\theta$$

are infinite for $k=1$. The third integral

$$\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\cos^2\theta}{\Delta}d\theta$$

is not infinite. The other three definite integrals of Legendre's third kind, involved in the question, are to the parameters,

$$n=-\frac{\sin\phi\sin\gamma}{\sin^2\frac{\phi+\gamma}{2}}, \quad n'=\frac{\sin\phi\sin\gamma}{\cos^2\frac{\phi+\gamma}{2}}, \quad m=-\frac{4ax'\sin\phi\sin\gamma}{(x'\sin\phi+a\sin\gamma)^2}.$$

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* Traité des Fonctions Elliptiques, Tom. I, p. 72.

The first integral is infinite when $\phi = \gamma$; the second when $\phi = \gamma$ and $x' = a$; the third when $\phi = \gamma$ and $x' = a$, or when $x' \sin\phi = a \sin\gamma$. When $x' = a$ the attracted point is on the sphere; if $f = x'\cos\phi - a\cos\gamma = 0$, it is on the plane of the base; if both conditions are fulfilled it is on the edge of the segment. The third condition $x' \sin\phi = a\sin\gamma$ has been introduced only by the elimination of the imaginary parameters. To make the discontinuities and indeterminations clear, it seems necessary to consider apart each case.

CASE I.—When $\phi = \gamma$.

Here we have

$$k^2 = \frac{4ax'\sin^2\gamma}{x'^2 - 2ax'\cos 2\gamma + a^2}, \quad g^2 = \frac{4a^2\sin^2\gamma}{x'^2 - 2ax'\cos 2\gamma + a^2},$$

and, putting

$$B = \frac{1}{3x'}(a^2 + 3ax' - 2x'^2) + \left(-\frac{2x'^2}{a} + 2x' - \frac{4}{3}a\right)\sin^2\gamma,$$

$$B' = 2\left(-x + \frac{2}{3}a\right)\sin^2\gamma,$$

$$B''' = -(x'+a)\left[1 - \frac{(2x'-a)(x'+a)^2}{6ax'^2}\sec.^2\gamma\right],$$

we have

$$X' = \mp\pi\rho a\left[1 + \frac{(2x'+a)(x'-a)^2}{3ax'^2}\right] + 2\rho g\cot\gamma\int_0^{\frac{\pi}{2}}\left[B + B'\sin^2\theta\right.$$
$$\left. + \frac{B'''}{1+\tan^2\gamma\sin^2\theta} + 2(x'-a)\sin^2\gamma\,\frac{\frac{k^2}{g^2} - \left[k'^2 - \frac{k^4}{g^2}\right]\sin^2\theta}{1-(g^2+k^2)\sin^2\theta + g^2\sin^4\theta}\right]\frac{d\theta}{\Delta}.$$

The upper of the ambiguous signs must be read when $x' - a$ is positive, the lower when it is negative.

CASE II.—When $x' = a$.

Here we have

$$k^2 = \frac{\sin\phi\sin\gamma}{\sin^2\frac{\phi+\gamma}{2}} = -n, \quad g^2 = \frac{\sin^2\gamma}{\sin^2\frac{\phi+\gamma}{2}},$$

and, putting

$$B = \left[\frac{1}{3} - 2\sin^2\phi\right]\cos\phi - \left[\frac{1}{3} - 2\sin^2\phi - \frac{2}{3}\cos(\phi+\gamma)\right]\cos\gamma,$$

$$B' = 2\left[-\cos\phi + \frac{2}{3}\cos\gamma\right]\sin\phi\sin\gamma,$$

$$B''' = 2\left[\frac{2}{3} - \cos^2\frac{\phi+\gamma}{2}\right]\frac{\cos\frac{\phi-\gamma}{2}}{\cos\frac{\phi+\gamma}{2}},$$

we have

$$X' = \mp\pi\rho a + \frac{2\rho g a}{\sin\gamma}\int_0^{\frac{\pi}{2}}\left[B + B'\sin^2\theta + \frac{B'''}{1 + n'\sin^2\theta}\right.$$
$$\left. + 2(\cos\phi - \cos\gamma)\sin\phi\sin\gamma\,\frac{\frac{k^2}{g^2} + \left[k'^2 - \frac{k^4}{g^2}\right]\sin^2\theta}{1-(g^2+k^2)\sin^2\theta + g^2\sin^4\theta}\right]\frac{d\theta}{\Delta}.$$

CASE III.—When $\phi = \gamma$ and $x' = a$.

Here we have

$$k^2 = 1, \quad g^2 = 1, \quad f = 0, \quad n' = \tan^2\gamma,$$

and, putting

$$B = \frac{2}{3} - \frac{4}{3}\sin^2\gamma, \quad B' = -\frac{2}{3}\sin^2\gamma, \quad B''' = -2\left(1 - \frac{2}{3}\sec^2\gamma\right),$$

we have

$$X' = \mp\pi\rho a + 2\rho a\cot\gamma\int_0^{\frac{\pi}{2}}\left[B + B'\sin^2\theta + \frac{B'''}{1 + \tan^2\gamma\sin^2\theta}\right]\frac{d\theta}{\cos\theta}.$$

In deriving this expression, it has been assumed that we might write $x - \mathrm{b}$ for x in a certain place, without thereby altering the value of the integral. But this is not permissible when the discontinuities have undergone a change. If we had written $x - a$ instead and neglected the discontinuity $\mp\pi\rho a$, the result would have been correct. To avoid the hap-hazard character of this course of proceeding, we return to the original expression for X, before integration by parts was employed.

Throwing out the factor $(a-x)^2$, common in this case to the numerators and denominators of the two terms, we have

$$X' = 2\rho\int_{\mathrm{b}}^{a}\left[-\text{arc sin}\sqrt{\frac{x-\mathrm{b}}{a+x}} + \sqrt{\frac{a-x}{2a}}\,\text{arc sin}\left(\frac{1}{\sin\gamma}\sqrt{\frac{x-\mathrm{b}}{a+x}}\right)\right]dx.$$

Employing integration by parts so as to bear on the arcs, we have, as the part freed from the sign of integration, the expression

$$-2\rho a \operatorname{arc} \sin \sqrt{\frac{a-\mathrm{b}}{2a}} + 2\rho \mathrm{b} \operatorname{arc} \sin \sqrt{\frac{\mathrm{b}-\mathrm{b}}{a+\mathrm{b}}}.$$

Now the second term of this vanishes except possibly when $a+\mathrm{b}=0$, or in the case when the segment is a hemisphere. But if the hemisphere has an attraction continuous with the attraction of the segments adjacent to it, it is plain we must assume that $\frac{\mathrm{b}-\mathrm{b}}{a+\mathrm{b}}=0$ even when $a+\mathrm{b}=0$. The value of the first term is $-2\rho a\gamma$, and, joining to this the portion still under the sign of integration, we have

$$X' = -2\rho a\gamma + \frac{\sqrt{2a}}{3}\rho \cos\gamma \int_{\mathrm{b}}^{a} \left[1 + \frac{a}{a+x}\right] \frac{dx}{\sqrt{x-\mathrm{b}}}.$$

Put $\sqrt{x-\mathrm{b}} = u$, then

$$\begin{aligned} X' &= -2\rho a\gamma + \tfrac{2}{3}\sqrt{2a}\rho \cos\gamma \int_0^{\sqrt{2a}\sin\gamma} \left[1 + \frac{a}{a+\mathrm{b}+u^2}\right] du \\ &= -2\rho a\gamma + \tfrac{2}{3}\sqrt{2a}\rho \cos\gamma \left[\sqrt{2a}\sin\gamma + \frac{a}{\sqrt{2a}\cos\gamma}\gamma\right] \\ &= \tfrac{4}{3}\rho a(\sin\gamma\cos\gamma - \gamma) \end{aligned}$$

In case $\gamma = \frac{\pi}{2}$, we have $X' = -\frac{2}{3}\pi\rho a$, which agrees with the known expression for the attraction of the homogeneous sphere for a point on its surface.

To determine the value of Y for the present case, we observe, that taking P for the independent variable, we have, for this case,

$$dx = 4a\sin^2\gamma \, . \, PdP;$$

thus

$$\begin{aligned} Y &= 4\rho a \sin^2\gamma \int_0^1 \left[\log\frac{1+P}{1-P} - 2P\right] PdP \\ &= 8\rho a \sin^2\gamma \int_0^1 \left[\frac{1}{3}P^4 + \frac{1}{5}P^6 + \frac{1}{7}P^8 + \ldots\right] dP. \\ &= 8\rho a \sin^2\gamma \quad \left[\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots\right]. \end{aligned}$$

The last factor of this is a convergent series, although it is the difference of two divergent series; and it is easy to see that its value is $\frac{1}{6}$. Then

$$Y = \frac{4}{3}\rho a \sin^2 \gamma.$$

Calling Y the lateral attraction, the formula shows that this is proportional to the square of the sine of the angular radius of the segment; and, in the case of the hemisphere, this quantity is obtained by dividing the downward attraction of the whole sphere by π.

CASE IV.—When $x' \sin \phi = a \sin \gamma$.

Here we have the multipliers of the infinite definite integral

$$1 - \frac{k^2}{g^2} = 0, \quad \sqrt{1 + m} = 0,$$

Therefore we must determine the limits towards which converge the two expressions

$$\int_0^{\frac{\pi}{2}} \frac{1 - \frac{k^2}{g^2}}{1 - \frac{4k^2 g^2}{(g^2 + k^2)^2} \sin^2 \theta} \frac{d\theta}{\Delta} \text{ and } \int_0^{\frac{\pi}{2}} \frac{\sqrt{1 + m}}{1 + m \sin^2 \theta} \frac{d\theta}{\Delta},$$

when m approaches -1. But we have, in this case,

$$1 - \frac{k^2}{g^2} = \pm\, 2\sqrt{1 + m},$$

positive if k^2 is smaller than g^2, and negative if it is greater. Hence we need consider only the latter expression. The radical $\sqrt{1 + m}$ being taken positively, at the limit $m = -1$, the value of the integral is $\frac{\pi}{2}\frac{1}{k'}$.[*] If k is unity this is infinite; but this case has already been treated.

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[*] This may easily be deduced from the equation (m') of Legendre, Traité des Fonctions Elliptiques, Tom. I, p. 138.